Discretization of 2^{nd} Order DE: Summary

Goal: Discretize 2^{nd} order equation on $[a,b]$ with h

$$y'' + p(x)y' + q(x)y = f(x)$$

$y(a) = c$ "fixed" $y(b) = d$

—or—

$y'(a) = c$ "free" $y'(b) = d$

Pointwise Method:

① Find sample points $x_0 = a,\ x_1 = a+h,\ \dots\ x_{n+1} = b$

② Write pointwise equations at $x_1, \dots x_n$

$$y_k'' + p(x_k)\,y_k' + q(x_k)\,y_k = f(x_k)$$

③ Substitute

(second diff) $y_k'' = \left(\tfrac{1}{h}\right)^2 \left(y_{k+1} - 2y_k + y_{k-1}\right)$

(centered diff) $y_k' = \dfrac{1/h}{2}\left(y_{k+1} - y_{k-1}\right)$

(boundary cond.) (In first & last equation)

$y(a) = c$ "fixed" $y(b) = d$

$\hookrightarrow y_0 = c$ $\hookrightarrow y_{n+1} = d$

$y'(a) = c$ "free" $y'(b) = d$

$\hookrightarrow y_0 = y_1 - h\cdot c$ $\hookrightarrow y_{n+1} = y_n + hd$

④ Simplify & convert to matrix equation

$$\begin{bmatrix} \text{Matrix} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} f(x_1) - \boxed{\cdots} \\ \vdots \\ f(x_n) - \boxed{\cdots} \end{bmatrix}$$

Matrix Operator Method:

① Find sample points $x_0 = a,\ x_1 = a+h,\ \dots,\ x_{n+1} = b$

② Convert parts of equation to matrices

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

mult. row by $p(x_k)$

$$\left(\begin{bmatrix} \text{second} \\ \text{diff. matrix} \end{bmatrix} + \begin{bmatrix} \text{centered} \\ \text{diff. mat.} \end{bmatrix} + \begin{bmatrix} q(x_k) \text{ on} \\ \text{diag. mat.} \end{bmatrix} \right) \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}$$

③ Account for boundary values $\begin{cases} f(x_1) \mapsto f(x_1) - \boxed{\cdots} \\ f(x_n) \mapsto f(x_n) - \boxed{\cdots} \end{cases}$

Note: Second difference & centered difference matrices change depending on type of boundary values.

y''

$$\left(\tfrac{1}{h}\right)^2 \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \qquad \left(\tfrac{1}{h}\right)^2 \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \qquad \left(\tfrac{1}{h}\right)^2 \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

fixed-fixed free-fixed free-free

y'

$$\frac{1/h}{2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \qquad \frac{1/h}{2} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \qquad \frac{1/h}{2} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

fixed-fixed free - fixed free-free

Note: Effect of boundary conditions:

$y(a) = c \rightsquigarrow y_0 = c$ vs. $y'(a) = c \rightsquigarrow y_0 = y_1 - h\cdot c$

$y(b) = d \rightsquigarrow y_{n+1} = d$ vs. $y'(b) = d \rightsquigarrow y_{n+1} = y_n + h\cdot d$

Ex: $y'' + 4y' + 5xy = 0$ on $[1,2]$ with $h = 1/5$ ($1/h = 5$)

with $y(1) = 0$ & $y(2) = 0$

$y(1) = 3$ & $y(2) = -2$ (in Red)

$$\left(25\begin{bmatrix}-2&1&0&0\\1&-2&1&0\\0&1&-2&1\\0&0&1&-2\end{bmatrix} + 4\cdot\frac{5}{2}\begin{bmatrix}0&1&0&0\\-1&0&1&0\\0&-1&0&1\\0&0&-1&0\end{bmatrix} + \begin{bmatrix}6&0&0&0\\0&7&0&0\\0&0&8&0\\0&0&0&9\end{bmatrix}\right)\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$25-10$, 3, 45; $25+10$, -2, (-70)

Top equation: $25(y_2 - 2y_1 + y_0) + 10(y_2 - y_0) + 6y_1 = 0$ ($y_0 = 3$, $y_0 = 3$)

$25(y_2 - 2y_1) + 10(y_2) + 6y_1 = 0 - 45$

Bottom equation: $25(y_5 - 2y_4 + y_3) + 10(y_5 - y_3) + 9y_4 = 0$ ($y_5 = -2$, $y_5 = -2$)

$25(-2y_4 + y_3) + 10(-y_3) + 9y_4 = 0 - (-70)$

$y'' + 4y' + 5xy = 0$ on $[1,2]$ with $h = 1/5$

with $y'(1) = 0$ & $y'(2) = 0$

$y'(1) = 3$ & $y'(2) = -2$ (in Red)

$$\left(25\begin{bmatrix}-1&1&0&0\\1&-2&1&0\\0&1&-2&1\\0&0&1&-1\end{bmatrix} + 4\cdot\frac{5}{2}\begin{bmatrix}-1&1&0&0\\-1&0&1&0\\0&-1&0&1\\0&0&-1&1\end{bmatrix} + \begin{bmatrix}6&0&0&0\\0&7&0&0\\0&0&8&0\\0&0&0&9\end{bmatrix}\right)\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$25-10$, $(-1/5)\cdot 3$, (-9); $25+10$, $(1/5)\cdot(-2)$, (-14)

free-free   free-free

Top eqn: $25(y_2 - 2y_1 + y_0) + 10(y_2 - y_0) + 6y_1 = 0$ ($y_0 = y_1 - 1/5\cdot 3$, $y_0 = y_1 - 1/5\cdot 3$)

$25(y_2 - y_1) + 10(y_2 - y_1) + 6y_1 = 0 - (-9)$

Bottom eqn: $25(y_5 - 2y_4 + y_3) + 10(y_5 - y_3) + 9y_4 = 0$ ($y_5 = y_4 + 1/5(-2)$, $y_5 = y_4 + 1/5(-2)$)

$25(-y_4 + y_3) + 10(y_4 - y_3) + 9y_4 = 0 - (-14)$